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Universal connections of elastic fibrous composites: some new results

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Abstract

Universal connections between the overall moduli of elastic fibrous composites are explored. For any medium that can be represented by certain characterization functions, we show that its effective modulus tensors follow similar constraints as those for Hill's connections for a two-phase fibrous composite. Some new standpoints are proposed, which reveal that the connections remain valid for media containing cavities or rigid inclusions. In addition, connections are devised to accomodate the case in which the composite consists of phases with identical eigenmoduli. We show that, in this particular case, it often provides additional constraints to the overall moduli of the composite. Specific results are given in analytic forms for two-phase fibrous composites with transversely isotropic phases, and with square-symmetric phases.

Keywords: Fibrous composites; Effective moduli; Universal connections

1. Introduction

In many physical problems it is permissible to set certain fields to be constant throughout a heterogeneous medium. This may be due to physical considerations or to geometrical arrangements. For instance, in a cylindrical body a constant axial strain or electric field may be prescribed or induced. In addition, in thermo-elastostatics one may assume that a uniform temperature change prevails in a solid. Quite a few exact theorems of composites are indeed a consequence of the existence of such constant fields. For example, Hill (1964) found that the overall elastic moduli of two-phase fibrous composites are connected by universal relations which are independent of the geometry at a given volume fraction. Levin (1967) showed that the effective mechanical properties and thermal expansion coefficient are

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related in an exact manner. Rosen and Hashin (1970) derived a relation between the specific heat and thermo-mechanical moduli. All these theorems are microstructure independent and provide theoretical linkages among the overall moduli of the composite. The results were originally presented in terms of isotropic or transversely isotropic elasticity, where explicit formulae can be found. In a series of works, Dvorak (1986, 1990) showed that, in the presence of a certain constant field, it is possible to generate uniform fields throughout the medium by a particular set of loadings. With this concept, much progress has been made in finding the connections between the moduli of composites with arbitrary anisotropy or with other physical context, such as piezoelectricity (see for example, Benveniste and Dvorak, 1992 and references cited therein). Recently, Chen (1998) showed that, upon a rearrangement of moduli, all the aforementioned connections are mathematically equivalent to each other and can be treated in a unified manner. However, the uniform field approach is not without limitation and the applicability of the connections may not seem readily apparent from the existing presentations. In fact, there are quite a few theoretically interesting situations in which the uniform field method may exhibit difficulty or even break down, for instance in porous media or in composites with identical bulk or shear moduli. This work proposes some new standpoints to these issues and intends to provide a solution to resolve the gap.

The formulation will focus exclusively on elastic cylindrical aggregates, which are sufficient to generate results in many different contexts, including thermal effects, humidity, electric fields, etc. Specifically, Section 2 attempts to show that the effective modulus tensors of a medium, that can be represented by certain characterization functions, follow similar forms of the universal connections for a two-phase fibrous composite. Particularly, the characterized material may even vary in space. The derivations are rather straightforward without invoking the uniform-field approach. Section 3 examines the situations in which a certain matrix is not invertible, which corresponds to the cases that uniform fields cannot be constructed throughout the body, in the way originally conceived. Interestingly, we observe that in this particular situation uniform fields can in fact always be constructed and that, in many situations, additional constraints to the overall moduli are thereby obtained. To our knowledge, this feature has not been noticed in any of the literature before. Two specific examples for two-phase composites, one for transversely isotropic phases and the other for square-symmetric phases, are worked out in detail. Exact connections between the overall moduli are given in analytic forms. Section 5 presents a general framework for the subject, which is suitable for different physical contexts. Central to the concept is the existence of certain constant quantities, e.g. temperature, axial strain, etc. Finally, some closing remarks are made in Section 6.

2. Universal connections of overall moduli

To illustrate the main concept of the framework, we shall focus on purely elastic behavior of fibrous composites. The concept will be sufficient to extend to other physical contexts, such as piezoelectricity, thermal effects, etc. On a fixed Cartesian coordinate system $\{x_i\}$, the constitutive equations for an elastic solid are given by $\sigma_{ij} = L_{ijkl}\varepsilon_{kl}$ or $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$. In matrix notation, they can be expressed as $\boldsymbol{\sigma} = \boldsymbol{L}\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are defined by

$$\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{31}, \quad \sigma_6 = \sigma_{12},$$

$$\varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = 2\varepsilon_{23}, \quad \varepsilon_5 = 2\varepsilon_{31}, \quad \varepsilon_6 = 2\varepsilon_{12}. \tag{1}$$

Suppose the axial direction is chosen as parallel to the x_3-axis. Then the usual Hooke's law can be rearranged as

$$\begin{bmatrix} \hat{\boldsymbol{\varepsilon}} \\ -\sigma_3 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{S}} & \boldsymbol{a} \\ \boldsymbol{a}^{\mathrm{T}} & -E_3 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\sigma}} \\ \varepsilon_3^0 \end{bmatrix}, \quad \begin{bmatrix} \hat{\boldsymbol{\sigma}} \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{L}} & \boldsymbol{b} \\ \boldsymbol{b}^{\mathrm{T}} & l_{33} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\varepsilon}} \\ \varepsilon_3^0 \end{bmatrix}, \tag{2}$$

where

$$\hat{\boldsymbol{\sigma}} = [\sigma_1, \sigma_2, \sigma_4, \sigma_5, \sigma_6]^{\mathrm{T}}, \quad \hat{\boldsymbol{\varepsilon}} = [\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5, \varepsilon_6]^{\mathrm{T}},$$
$$\boldsymbol{a} = [s_{13}, s_{23}, s_{34}, s_{35}, s_{36}]^{\mathrm{T}}/s_{33}, \quad \boldsymbol{b} = [l_{13}, l_{23}, l_{34}, l_{35}, l_{36}]^{\mathrm{T}},$$
$$(\hat{\boldsymbol{L}})_{ij} = l_{ij}, \quad (\hat{\mathbf{S}})_{ij} = s_{ij} - s_{i3}s_{j3}/s_{33}, \quad (i,j = 1, 2, 4, 5, 6), \tag{3}$$

E_3 is the Young's modulus in the x_3-direction, and s_{ij} and l_{ij} are the usual two-index compliance and stiffness, respectively. The superscript 'T' denotes the matrix transpose. The field quantities must satisfy the equilibrium equations and compatibility conditions. Along any interfaces of different materials, perfect bonding is assumed. For brevity, in the sequel, the formulation is derived based on $(2)_1$. The dual approach via the moduli is obtained simply by letting $\hat{\boldsymbol{L}} \leftrightarrow \hat{\boldsymbol{S}}$, $\hat{\boldsymbol{b}} \leftrightarrow \hat{\boldsymbol{a}}$, $l_{33} \leftrightarrow -E_3$. In the sequel, 'phase', as the usual interpretation, is defined as a region with constant moduli; 'a pointwise varying material' means a material which has moduli depending on positions.

Let us now consider a heterogeneous medium consisting of cylindrical phases with arbitrary transverse geometry. Suppose that the properties of the medium can be represented by the forms

$$\hat{\boldsymbol{S}}(x_1, x_2) = \hat{\boldsymbol{S}}_\alpha + \left(\hat{\boldsymbol{S}}_\beta - \hat{\boldsymbol{S}}_\alpha\right)\boldsymbol{F}(x_1, x_2),$$
$$\boldsymbol{a}(x_1, x_2) = \boldsymbol{a}_\alpha + \boldsymbol{F}^{\mathrm{T}}(x_1, x_2)(\boldsymbol{a}_\beta - \boldsymbol{a}_\alpha),$$
$$E_3(x_1, x_2) = E_3^\alpha + \left(E_3^\beta - E_3^\alpha\right)f(x_1, x_2), \tag{4}$$

in which $\{\hat{\boldsymbol{S}}_i, \boldsymbol{a}_i, E_3^i\}$, $i = \alpha, \beta$, are two set of constant properties. $\boldsymbol{F}(x_1, x_2)$ and $f(x_1, z_2)$ are certain functions of position restricted by the requirements that the compliances be symmetric and non-negative. The objective of this section is to show that the effective moduli of the considered medium will follow the universal connections as those for a two-phase fibrous composite.

Equation (4) depicts a wide class of heterogeneous media. For example, for any two-phase medium consisting of phases 1 and 2, the properties of the medium can be characterized by selecting $\hat{\boldsymbol{S}}_\alpha = \hat{\boldsymbol{S}}_1$, $\hat{\boldsymbol{S}}_\beta = \hat{\boldsymbol{S}}_2$, $E_3^\alpha = E_3^1$, $E_3^\beta = E_3^2$ so that $\boldsymbol{F}(\boldsymbol{x})$ and $f(\boldsymbol{x})$ follow

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{cases} 0 & \text{for: } \boldsymbol{x} \in \Omega_1 \\ \boldsymbol{I} & \text{for: } \boldsymbol{x} \in \Omega_2 \end{cases}, \quad f(\boldsymbol{x}) = \begin{cases} 0 & \text{for: } \boldsymbol{x} \in \Omega_1 \\ 1 & \text{for: } \boldsymbol{x} \in \Omega_2 \end{cases}, \tag{5}$$

and $\boldsymbol{a}_\alpha = \boldsymbol{a}_1, \boldsymbol{a}_\beta = \boldsymbol{a}_2$, where $\boldsymbol{I}$ is the 5×5 unit matrix and, Ω_1 and Ω_2 denote the domains of phases 1 and 2, respectively. Of course, this representation is not a unique choice. In particular, for any given two-phase medium one can always select $\hat{\boldsymbol{S}}_\alpha, \hat{\boldsymbol{S}}_\beta, E_3^\alpha$ and E_3^β at will, while $\boldsymbol{a}_i, \boldsymbol{F}_i$ and f_i are restricted by some relations. In other words, there are a number of ways to represent the elastic properties of a two-phase medium via (4). A notable feature of (4) is that it is capable of modeling a specific class of inhomogeneous materials. Particularly, the characterized material properties may even vary in space since $\boldsymbol{F}$ and f are functions of position. We note, however, that for an arbitrary three- or multi-phase medium it is not always possible to characterize the material properties via (4), since $\hat{\boldsymbol{S}}$ and $\boldsymbol{a}$ are linked by $\boldsymbol{F}$. But for a three-phase material in which the third phase is itself a composite of the first two materials, it is indeed possible to characterize the medium by (4). Lastly, it can be verified that if $(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1)$ is singular so is $(\boldsymbol{F}_2 - \boldsymbol{F}_1)$, since one can choose $(\hat{\boldsymbol{S}}_\beta - \hat{\boldsymbol{S}}_\alpha)$ arbitrarily. The same reasoning also applies to E_3. Eqn (4) also implies that $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}_\beta$ cannot be uniquely determined if $(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1)$ is not invertible.

Suppose, under a uniform loading, the macroscopic behavior of the medium can be effectively represented by the same constitutive relation (2) with certain suitably chosen overall moduli $\hat{\boldsymbol{S}}^*$, $\boldsymbol{a}^*$ and E_3^*. Then the average fields satisfy

$$\begin{bmatrix} \langle \hat{\boldsymbol{\varepsilon}} \rangle \\ -\langle \sigma_3 \rangle \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{S}}^* & \boldsymbol{a}^* \\ \boldsymbol{a}^{*\mathrm{T}} & -E_3^* \end{bmatrix} \begin{bmatrix} \langle \hat{\boldsymbol{\sigma}} \rangle \\ \varepsilon_3^0 \end{bmatrix}, \tag{6}$$

where the argument inside $\langle\ \rangle$ means the volume averages over the representative volume element (RVE). By taking the average of (2) and comparing with (6) it follows that

$$\begin{aligned} &\Delta\hat{\boldsymbol{S}}^* \langle \hat{\boldsymbol{\sigma}} \rangle + \Delta\boldsymbol{a}^* \varepsilon_3^0 = \Delta\hat{\boldsymbol{S}} \langle \boldsymbol{F}\hat{\boldsymbol{\sigma}} \rangle + \langle \boldsymbol{F}^{\mathrm{T}} \rangle \Delta\boldsymbol{a}\varepsilon_3^0, \\ &\Delta\boldsymbol{a}^{*\mathrm{T}} \langle \hat{\boldsymbol{\sigma}} \rangle - \Delta E_3^* \varepsilon_3^0 = \Delta\boldsymbol{a}^{\mathrm{T}} \langle \boldsymbol{F}\hat{\boldsymbol{\sigma}} \rangle - \Delta E_3 \langle f \rangle \varepsilon_3^0, \end{aligned} \tag{7}$$

where

$$\begin{aligned} &\Delta\hat{\boldsymbol{S}}^* = \hat{\boldsymbol{S}}^* - \hat{\boldsymbol{S}}_\alpha, \quad \Delta\boldsymbol{a}^* = \boldsymbol{a}^* - \boldsymbol{a}_\alpha, \quad \Delta E_3^* = E_3^* - E_3^\alpha, \\ &\Delta\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_\beta - \hat{\boldsymbol{S}}_\alpha, \quad \Delta\boldsymbol{a} = \boldsymbol{a}_\beta - \boldsymbol{a}_\alpha, \quad \Delta E_3 = E_3^\beta - E_3^\alpha. \end{aligned} \tag{8}$$

Since $\langle \hat{\boldsymbol{\sigma}} \rangle$ and ε_3^0 can be prescribed arbitrarily, by letting $\varepsilon_3^0 = 0$, relations (9) provide

$$\Delta\boldsymbol{a}^* = \Delta\hat{\boldsymbol{S}}^* \Delta\hat{\boldsymbol{S}}^{-1} \Delta\boldsymbol{a}, \tag{9}$$

while, letting $\langle \hat{\boldsymbol{\sigma}} \rangle = 0$, they lead to

$$-\left[\Delta E_3^* - \langle f \rangle \Delta E_3\right] = \Delta\boldsymbol{a}^{\mathrm{T}} \Delta\hat{\boldsymbol{S}}^{-1} \left[\Delta\boldsymbol{a}^* - \langle \boldsymbol{F}^{\mathrm{T}} \rangle \Delta\boldsymbol{a}\right]. \tag{10}$$

Eqns (9) and (10) provide at most six constraints to the overall properties of the non-homogeneous medium, which means that the six material constants relevant to the axial direction of the cylindrical aggregate can be solely determined by the remaining 15 off-axial material constants. In particular, the connection (9) suggests that the overall moduli of a family of heterogeneous materials are governed by the same constraints as that for a typical two-phase fibrous composite. In addition, the relationship (9) is independent of volume concentrations of the constituents. The functions $\boldsymbol{F}$ and f, which are relevant to the volume concentrations of the phases, only take effect in one connection (10).

For the usual two-phase composite (5), it follows that $\langle \boldsymbol{F} \rangle = c_2\boldsymbol{I}$ and $\langle f \rangle = c_2$, where c_2 denotes the volume fraction of phase 2. Eqns (9) and (10) are recast as

$$\begin{aligned} &(\boldsymbol{a}^* - \boldsymbol{a}_1) = \left(\hat{\boldsymbol{S}}^* - \hat{\boldsymbol{S}}_1\right)\left(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1\right)^{-1}(\boldsymbol{a}_2 - \boldsymbol{a}_1), \\ &-\left(E_3^* - c_1E_3^1 - c_2E_3^2\right) = (\boldsymbol{a}_2 - \boldsymbol{a}_1)^{\mathrm{T}}\left(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1\right)^{-1}(\boldsymbol{a}^* - c_1\boldsymbol{a}_1 - c_2\boldsymbol{a}_2), \end{aligned} \tag{11}$$

which are exactly the results of Chen [1998, eqns $(27)_1$ and $(28)_1$] derived from the uniform-field approach. The present formulation does not invoke the concept of uniform fields and yet the scope is somewhat broader than that of the previous works. For example, the present approach justifies the validity of the connections for media containing cavities or rigid inclusions, and for a few classes of pointwise varying materials, in which uniform fields may not be generated. We note, however, that for a general three- or more-phase material, no such connections (9) and (10), can be found by either approach. As noted earlier by Chen (1998), the connections are formally identical with Levin's (1967) relation, Rosen and Hashin's (1970) connection between the effective thermal properties and overall mechanical properties, and with the restrictions of the overall electrical–mechanical coupling behavior, etc. Thus, by proper interpretations of the definitions in (9) and (10), the results also apply to the constraints of overall moduli of various physical phenomena.

Before closing this section, we ask whether the choices of $\hat{\boldsymbol{S}}$ will affect the results (9) and (10)? In other words, if two sets of $\hat{\boldsymbol{S}}_\alpha$, $\hat{\boldsymbol{S}}_\beta$, E_3^α, E_3^β, $(\boldsymbol{a}_\alpha, \boldsymbol{a}_\beta, \boldsymbol{F})$ represent the same configurations of a medium, will the connections obtained remain the same? The answer is yes. To prove this, we start from (9). Referring to (9) and (8), it is seen that

$$\boldsymbol{a}^* = \left(\boldsymbol{a}_\alpha - \hat{\boldsymbol{S}}_\alpha \Delta \hat{\boldsymbol{S}}^{-1} \Delta \boldsymbol{a}\right) + \hat{\boldsymbol{S}}^* \left(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1\right)^{-1} (\boldsymbol{a}_2 - \boldsymbol{a}_1) \tag{12}$$

$$= \boldsymbol{a}_1 - \hat{\boldsymbol{S}}_1 \left(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1\right)^{-1} (\boldsymbol{a}_2 - \boldsymbol{a}_1) + \hat{\boldsymbol{S}}^* \left(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1\right)^{-1} (\boldsymbol{a}_2 - \boldsymbol{a}_1). \tag{13}$$

Similarly, since $\langle \boldsymbol{F}^{\mathrm{T}} \rangle \Delta \boldsymbol{a} = \langle \boldsymbol{a}(\boldsymbol{x}) \rangle - \boldsymbol{a}_\alpha$ and $\langle f \rangle \Delta E_3 = \langle E_3(\boldsymbol{x}) \rangle - E_3^\alpha$, it turns out that (10) can be recast as

$$-\left(E_3^* - \langle E_3(\boldsymbol{x}) \rangle\right) = \Delta \boldsymbol{a}^{\mathrm{T}} \Delta \hat{\boldsymbol{S}}^{-1} \left(\boldsymbol{a}^* - \langle \boldsymbol{a}(\boldsymbol{x}) \rangle\right). \tag{14}$$

Obviously, (9) and (10) are independent of the choices of $\hat{\boldsymbol{S}}_\alpha$ and $\hat{\boldsymbol{S}}_\beta$.

3. Universal connections when $\Delta \hat{S}$ is singular

Returning to the connections (11) for a two-phase fibrous composite, it is essential that $(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1)$ be invertible. As mentioned before, eqn (4) also implies that $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}_\beta$ cannot be uniquely determined if $(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1)$ is not invertible. Thus, the previous results (9) and (10), are not applicable when $(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1)$ is singular. Although it rarely occurs in practice that this matrix is singular, there are quite a few theoretically interesting outcomes resulting from this coincidence, for example in two-phase isotropic media with equal shear rigidities or equal bulk moduli. In this section we intend to prove that, with the concept of uniform fields, quite a few connections between the overall moduli can still be established.

For simplicity, let us focus on two-phase fibrous composites. The phase properties and overall moduli are written by (2) and (6), with indices $i = 1$, 2 and *, respectively. Suppose the composite aggregate is subjected to a uniform stress $\hat{\sigma}$ together with a certain constant axial strain ε_3^0 such that the strain $\hat{\boldsymbol{\varepsilon}}$ is constant throughout the whole medium, namely

$$\left(\hat{\boldsymbol{S}}_2 - \hat{\boldsymbol{S}}_1\right) \hat{\boldsymbol{\sigma}} + (\boldsymbol{a}_2 - \boldsymbol{a}_1) \varepsilon_3^0 = 0. \tag{15}$$

Since the stress and strain are constant throughout, the equilibrium and compatibility equations are automatically satisfied. Now suppose the medium is effectively represented by a homogeneous medium with certain unknown overall moduli. In other words, when the loads $\hat{\boldsymbol{\sigma}}$ and ε_3^0 are applied, the induced strain will be identical with the pointwise local strain, which means that

$$\left(\hat{\boldsymbol{S}}^* - \hat{\boldsymbol{S}}_i\right) \hat{\boldsymbol{\sigma}} + (\boldsymbol{a}^* - \boldsymbol{a}_i) \varepsilon_3^0 = 0, \quad i = 1, 2 \tag{16}$$

in the transverse direction, and

$$\left[\boldsymbol{a}^{*\mathrm{T}}-\sum_{r=1}^{2}c_r\boldsymbol{a}_r^{\mathrm{T}}\right]\hat{\boldsymbol{\sigma}}-\left[E_3^*-\sum_{r=1}^{2}c_rE_3^r\right]\varepsilon_3^0=0 \tag{17}$$

in the axial direction.

Back to (15), since $(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)$ is singular, the rank of $(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)$ must be less than five, say r. The non-trivial solutions of (15) can be grouped into the following cases:

- when rank $[(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)\,|\,(\boldsymbol{a}_2-\boldsymbol{a}_1)]=r+1<5$, there exist $(4-r)$ linearly independent vectors $[\tilde{\boldsymbol{\sigma}}|\varepsilon_3^0]^{\mathrm{T}}$ or $(5-r)$ linearly independent vector $[\tilde{\boldsymbol{\sigma}}|0]^{\mathrm{T}}$;
- when rank $[(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)|(\boldsymbol{a}_2-\boldsymbol{a}_1)]=5$, only one solution $[\tilde{\boldsymbol{\sigma}}|0]^{\mathrm{T}}$ can be found;
- when rank $[(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)|(\boldsymbol{a}_2-\boldsymbol{a}_1)]=r$, then there exist $(5-r)$ linearly independent vectors $[\hat{\boldsymbol{\sigma}}|\varepsilon_3^0]^{\mathrm{T}}$.

Here, the symbol $[\hat{\boldsymbol{S}}|\boldsymbol{a}]$ denotes the augmented matrix of $\hat{\boldsymbol{S}}$ and $\boldsymbol{a}$. Note that these non-trivial solutions are exactly the loads that generate the uniform fields throughout the medium. Substituting the solutions of $[\tilde{\boldsymbol{\sigma}}|\varepsilon_3^0]^{\mathrm{T}}$ into the identities (16) and (17) will provide the connections between the overall moduli for the case that the matrix $(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)$ is singular. It should be noted that, compared to previous results, for the most general anisotropic composite, the results may provide more than six constraints, since at least one linearly independent solution can be found. Of course the actual reduction must be determined for each particular system.

In view of (9) and (10) it seems that the connections may break down when the matrix $(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)$ is not invertible. However, the present derivations indicate that in this particular situation uniform fields can always be generated under certain loadings, even for a medium which is not a cylindrical aggregate. In the next section, we shall work out a few specific examples for a two-phase fibrous composite.

4. Example

In this section we derive some explicit formulae for the constraints between the overall moduli, in the case that the matrix $(\hat{\boldsymbol{S}}_2-\hat{\boldsymbol{S}}_1)$ is not invertible. To start the formulation, let us introduce the orthonormal basis

$$\boldsymbol{e}_0=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \boldsymbol{e}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \boldsymbol{e}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1\\1&0\end{bmatrix}, \tag{18}$$

in the symmetric second-order tensor space. It is known (Zheng and Hwang, 1996) that the general anisotropic in-plane compliance can be expressed as

$$\hat{S}=\frac{1}{E}\left[\underbrace{(1+v)\mathbf{I}-v\mathbf{1}\otimes\mathbf{1}}_{\text{isotropic part}}+\underbrace{\tfrac{1}{2}(\mathbf{1}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{1})+\mathbf{D}}_{\text{anisotropic part}}\right], \tag{19}$$

where E is the (in-plane) two-dimensional Young's modulus, v is Poisson's ratio in the transverse plane, $\mathbf{I}$ is the fourth-order identity tensor, $\mathbf{1}$ is the unit second-order tensor, and $\mathbf{d}=d_1\boldsymbol{e}_1+d_2\boldsymbol{e}_2$, $\mathbf{D}=D_1(\boldsymbol{e}_1\otimes\boldsymbol{e}_1-\boldsymbol{e}_2\otimes\boldsymbol{e}_2)+D_2(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1)$. The first brace in (19) indicates the isotropic part of the compliance tensor which is invariant to rotations with respect to the x_3-axis, while the second brace represents the anisotropic part of the tensor. In the basis (18) the constitutive equations in (2) can be written in matrix form as

$$\begin{bmatrix} (\varepsilon_1+\varepsilon_2)/\sqrt{2} \\ (\varepsilon_1-\varepsilon_2)/\sqrt{2} \\ \varepsilon_6/\sqrt{2} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1-v & d_1 & d_2 \\ d_1 & 1+v+D_1 & D_2 \\ d_2 & D_2 & 1+v-D_1 \end{bmatrix}\begin{bmatrix} (\sigma_1+\sigma_2)/\sqrt{2} \\ (\sigma_1-\sigma_2)/\sqrt{2} \\ \sqrt{2}\sigma_6 \end{bmatrix} + \varepsilon_3^0\begin{bmatrix} (a_1+a_2)/\sqrt{2} \\ (a_1-a_2)/\sqrt{2} \\ a_6/\sqrt{2} \end{bmatrix}, \quad (20)$$

$$-\sigma_3 = \frac{(a_1+a_2)}{\sqrt{2}}\frac{(\sigma_1+\sigma_2)}{\sqrt{2}} + \frac{(a_1-a_2)}{\sqrt{2}}\frac{(\sigma_1-\sigma_2)}{\sqrt{2}} + \frac{a_6}{\sqrt{2}}\frac{\sigma_6}{\sqrt{2}} - E_3\varepsilon_3^0. \quad (21)$$

There are four kinds of elastic symmetries corresponding to the in-plane compliance tensor $\hat{\boldsymbol{S}}$, namely full anisotropy, orthotropy, square symmetry and isotropy. Positive definiteness of the elastic matrix requires that

$$\det\hat{\boldsymbol{S}} > 0, \quad 1-v > 0, \quad 1+v+D_1 > 0, \quad 1+v-D_1 > 0,$$
$$d_1^2 < (1-v)(1+v+D_1), \quad d_2^2 < (1-v)(1+v-D_1), \quad D_2^2 < (1+v)^2 - D_1^2. \quad (22)$$

It is known that for an isotropic solid the area bulk modulus k and the shear modulus μ can be written as $k = E/2(1-v)$, $\mu = E/2(1+v)$. An advantage of the orthonormal basis is that when the axes are rotated around the x_3-axis by an angle of θ, the rotated compliance simply follows $\hat{\boldsymbol{S}}_\theta = \mathscr{R}\hat{\boldsymbol{S}}\mathscr{R}^{\mathrm{T}}$, in which

$$\mathscr{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{bmatrix},$$

$$\hat{\boldsymbol{S}}_\theta = \frac{1}{E}\begin{bmatrix} 1-v & d_1\cos 2\theta + d_2\sin 2\theta & -d_1\sin 2\theta + d_2\cos 2\theta \\ & 1+v+D_1\cos 4\theta + D_2\cos 4\theta & -D_1\sin 4\theta + D_2\cos 4\theta \\ \text{sym} & & 1+v-D_1\cos 4\theta - D_2\cos 4\theta \end{bmatrix}, \quad (23)$$

4.1. Transversely isotropic phases

Let us consider a two-phase fibrous composite in which the constituents are transversely isotropic. In this case, it is known that $d_1 = d_2 = D_1 = D_2 = a_6 = 0, a_1 = a_2 = -v^L$, v^L being the axial Poisson's ratio. For convenience, the indices i and m are used to distinguish the phases. For the case that $\Delta\hat{\boldsymbol{S}}$ is invertible, Hill (1964) showed that there exist two constraints between the overall properties of the composite. However, if the matrix $(\hat{\boldsymbol{S}}_i - \hat{\boldsymbol{S}}_m)$ is singular, the validity of the connections needs to be further examined. To explore this, by demanding $[\hat{\boldsymbol{S}}_i - \hat{\boldsymbol{S}}_m]\hat{\boldsymbol{\sigma}} + (\boldsymbol{a}_i - \boldsymbol{a}_m)\varepsilon_3^0 = 0$, namely

$$\frac{1}{2}\begin{bmatrix} \frac{1}{k_i}-\frac{1}{k_m} & 0 & 0 \\ 0 & \frac{1}{\mu_i}-\frac{1}{\mu_m} & 0 \\ 0 & 0 & \frac{1}{\mu_i}-\frac{1}{\mu_m} \end{bmatrix}\begin{Bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_6 \end{Bmatrix} - \sqrt{2}\varepsilon_3^0\begin{bmatrix} v_i^L - v_m^L \\ 0 \\ 0 \end{bmatrix} = 0, \quad (24)$$

one finds a particular loading path that generates uniform fields throughout the domain. It is then clear that the matrix $(\hat{\mathbf{S}}_i - \hat{\mathbf{S}}_m)$ singular implies $k_i = k_m$ or $\mu_i = \mu_m$. When the bulk moduli are equal, the null space of the system is one-dimensional and the only non-trivial solution is $[p, 0, 0, 0]^T$; when the shear rigidities are equal, the null space becomes two-dimensional and the solutions may be written as $[0, p, q, 0]^T$ or $[2\sqrt{2}(v_i^L - v_m^L)(k_i^{-1} - k_m^{-1})^{-1}, p, q, 1]^T$, p, q being some arbitrary constants. Recalling that the overall properties must be connected by (16) and (17), one finds that if $k_i = k_m$, then

$$k^* = k_i = k_m, \quad v_*^L = c_i v_i^L + c_m v_m^L; \tag{25}$$

if $\mu_i = \mu_m$, then

$$\mu^* = \mu_i = \mu_m, \quad \frac{v_*^L - v_m^L}{\dfrac{1}{k^*} - \dfrac{1}{k_m}} = -4\frac{v_*^L - c_i v_i^L - c_m v_m^L}{E^* - c_i E_i - c_m E_m} = \frac{v_i^L - v_m^L}{\dfrac{1}{k_i} - \dfrac{1}{k_m}}. \tag{26}$$

4.2. Square-symmetric phases

To further illustrate the procedures described in Section 3, we consider a two-phase fibrous medium in which the phase properties are of square symmetry (Hahn, 1987), namely $d_1 = d_2 = 0$ and $D_1^2 + D_2^2 \neq 0$ in (20). In particular, in the basis (20) the non-zero coefficients of $\hat{\mathbf{S}}$ are $\hat{\mathbf{S}}_{11} = \hat{s}_{11} + \hat{s}_{12}$, $\hat{\mathbf{S}}_{22} = \hat{s}_{11} - \hat{s}_{12}$, $\hat{\mathbf{S}}_{23} = \hat{\mathbf{S}}_{32} = \hat{s}_{16}$ and $\hat{\mathbf{S}}_{33} = \hat{s}_{66}/2$. It is mentioned that square symmetry is a two-dimensional version of tetragonal symmetry (Nye, 1985). One of the main differences with the transverse isotropy is that the material principal axes may vary with the phases. For fixed Cartesian coordinates, on the basis (18), the phase moduli can be written as

$$\hat{\mathbf{S}}_m = \frac{1}{E_m}\begin{bmatrix} 1 - v_m & 0 & 0 \\ 0 & 1 + v_m + D_1^m & D_2^m \\ 0 & D_2^m & 1 + v_m - D_1^m \end{bmatrix},$$

$$\mathbf{a}_m = \sqrt{2}\begin{bmatrix} s_{31}^m / s_{33}^m \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{\mathbf{S}}_i = \frac{1}{E_i}\begin{bmatrix} 1 - v_i & 0 & 0 \\ 0 & 1 + v_i + D_1^i & D_2^i \\ 0 & D_2^i & 1 + v_i - D_1^i \end{bmatrix},$$

$$\mathbf{a}_i = \sqrt{2}\begin{bmatrix} s_{31}^i / s_{33}^i \\ 0 \\ 0 \end{bmatrix}, \tag{27}$$

and the difference of the moduli $(\hat{\mathbf{S}}_i - \hat{\mathbf{S}}_m)$ follows as

$$\hat{\boldsymbol{S}}_i - \hat{\boldsymbol{S}}_m = \frac{1}{E_m}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta + \tilde{D}_1 & \tilde{D}_2 \\ 0 & \tilde{D}_2 & \beta - \tilde{D}_1 \end{bmatrix}, \tag{28}$$

where

$$\alpha = (1 - v_i)\eta - (1 - v_m), \quad \beta = (1 + v_i)\eta - (1 + v_m),$$
$$\tilde{D}_1 = D_1^i \eta - D_1^m, \quad \tilde{D}_2 = D_2^i \eta - D_2^m, \quad \eta = E_m/E_i. \tag{29}$$

By the relation (23), it is seen that the matrix can always be diagonalized via a rotation about the x_3-axis by an angle θ, which satisfies $\tilde{D}_1 \sin 4\theta = \tilde{D}_2 \cos 4\theta$. As in (15), one can find a particular loading that generates uniform fields in the medium by demanding $\mathscr{R}(\hat{\boldsymbol{S}}_i - \hat{\boldsymbol{S}}_m)\mathscr{R}^{\mathrm{T}}\mathscr{R}\hat{\boldsymbol{\sigma}} + \mathscr{R}(\boldsymbol{a}_i - \boldsymbol{a}_m)\varepsilon_3^0 = 0$, which provides

$$\frac{1}{E_m}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta + D_1^\theta & 0 \\ 0 & 0 & \beta - D_1^\theta \end{bmatrix}\begin{Bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_6 \end{Bmatrix} + \sqrt{2}\varepsilon_3^0\begin{bmatrix} \Delta \\ 0 \\ 0 \end{bmatrix} = 0, \tag{30}$$

where $D_1^\theta = \tilde{D}_1 \cos 4\theta - \tilde{D}_2 \sin 4\theta$, $\Delta = s_{31}^i/s_{33}^i - s_{31}^m/s_{33}^m$ and $\tilde{\boldsymbol{\sigma}} = \mathscr{R}\hat{\boldsymbol{\sigma}}$.

That the matrix $(\hat{\boldsymbol{S}}_i - \hat{\boldsymbol{S}}_m)$ is singular implies that $\alpha = 0$ and/or $\beta^2 - \tilde{D}_1^2 - \tilde{D}_2^2 = 0$. Specifically, the non-trivial solutions of eqn (30) can be found as

$$\begin{aligned}
&(a)\ \text{when}\ \alpha = 0 \Longrightarrow \left[\tilde{\boldsymbol{\sigma}}, \varepsilon_3^0\right] = [p, 0, 0, 0], \\
&(b)\ \text{when}\ \beta + D_1^\theta = 0 \Longrightarrow \left[\tilde{\boldsymbol{\sigma}}, \varepsilon_3^0\right] = \left[\sqrt{2}p\Delta, q, 0, -p\alpha/E_m\right], \\
&(c)\ \text{when}\ \beta - D_1^\theta = 0 \Longrightarrow \left[\tilde{\boldsymbol{\sigma}}, \varepsilon_3^0\right] = \left[\sqrt{2}p\Delta, 0, q, -p\alpha/E_m\right], \\
&(d)\ \text{when}\ \alpha = 0 \quad \text{and} \quad \beta + D_1^\theta = 0 \Longrightarrow \left[\tilde{\boldsymbol{\sigma}}, \varepsilon_3^0\right] = [p, q, 0, 0], \\
&(e)\ \text{when}\ \alpha = 0 \quad \text{and} \quad \beta - D_1^\theta = 0 \Longrightarrow \left[\tilde{\boldsymbol{\sigma}}, \varepsilon_3^0\right] = [p, 0, q, 0],
\end{aligned} \tag{31}$$

p, q being arbitrary constants. Note that, except for the first case that represents the identical plane strain bulk moduli in both phases, the null spaces of the non-trivial solutions are all two-dimensional. In either case, the overall moduli of the composite must comply with the connections (16) and (17), which now take the forms

$$\left(\hat{\boldsymbol{S}}^* - \hat{\boldsymbol{S}}_m\right)\mathscr{R}^{\mathrm{T}}\tilde{\boldsymbol{\sigma}} + (\boldsymbol{a}^* - \boldsymbol{a}_m)\varepsilon_3^0 = 0, \tag{32}$$

$$\left[\boldsymbol{a}^{*\mathrm{T}} - \sum_{r=i}^{m} c_r \boldsymbol{a}_r^{\mathrm{T}}\right]\mathscr{R}^{\mathrm{T}}\tilde{\boldsymbol{\sigma}} - \left[E_3^* - \sum_{r=i}^{m} c_r E_3^r\right]\varepsilon_3^0 = 0, \tag{33}$$

where

$$\hat{\boldsymbol{S}}^* - \hat{\boldsymbol{S}}_m = \frac{1}{E_m}\begin{bmatrix} \alpha^* & 0 & 0 \\ 0 & \beta^* + \tilde{D}_1^* & \tilde{D}_2^* \\ 0 & \tilde{D}_2^* & \beta^* - \tilde{D}_1^* \end{bmatrix}, \quad \boldsymbol{a}^* - \boldsymbol{a}_m = \sqrt{2}\begin{bmatrix} \Delta^* \\ 0 \\ 0 \end{bmatrix}, \tag{34}$$

and

$$\eta^* = E_m/E^*, \quad \alpha^* = (1-\nu^*)\eta^* - (1-\nu_m), \quad \beta^* = (1+\nu^*)\eta^* - (1+\nu_m),$$

$$\tilde{D}_1^* = D_1^*\eta^* - D_1^m, \quad \tilde{D}_2^* = D_2^*\eta^* - D_2^m, \quad \Delta^* = s_{31}^*/s_{33}^* - s_{31}^m/s_{33}^m. \tag{35}$$

The substitution of (31) into (33) will provide the constraints between the overall moduli of the composite. Specifically, in reference to (31), we find that the connections between the overall moduli of the composite are:

Case (a):

$$k^* = k_m = k_i, \quad s_{31}^*/s_{33}^* = c_i s_{31}^i/s_{33}^i + c_m s_{31}^m/s_{33}^m, \tag{36}$$

Case (b):

$$\frac{s_{31}^*/s_{33}^* - c_i s_{31}^i/s_{33}^i - c_m s_{31}^m/s_{33}^m}{E_3^* - c_i E_3^i - c_m E_3^m} = -4(1/k_i - 1/k_m)/\left(s_{31}^i/s_{33}^i - s_{31}^m/s_{33}^m\right), \tag{37}$$

$$\alpha^*/\alpha = \Delta^*/\Delta = \left(\frac{1}{k^*} - \frac{1}{k_m}\right) \Big/ \left(\frac{1}{k_i} - \frac{1}{k_m}\right), \tag{38}$$

$$\left(\beta^* + \tilde{D}_1^*\right)\cos 2\theta + \tilde{D}_2^* \sin 2\theta = 0, \quad \tilde{D}_2^* \cos 2\theta + \left(\beta^* - \tilde{D}_1^*\right)\sin 2\theta = 0, \tag{39}$$

Case (c): eqns (37), (38) and

$$-\left(\beta^* + \tilde{D}_1^*\right)\sin 2\theta + \tilde{D}_2^* \cos 2\theta = 0, \quad -\tilde{D}_2^* \sin 2\theta + \left(\beta^* - \tilde{D}_1^*\right)\cos 2\theta = 0, \tag{40}$$

Case (d): eqns (36) and (39),

Case (e): eqns (36) and (40).

5. General framework

In Section 2 we focused on the constraints between the overall elastic moduli of fibrous composites, in which the axial strain ε_3 can be taken constant. In fact, as long as certain local fields are uniform throughout the medium, the framework remains valid. For example, in a cylindrical electro-elastic aggregate the axial strain and axial electric field may be taken constant; also in a layered medium the strains in the transverse direction can be assumed uniform under certain loadings. Apart from these, uniform eigenfields, or transformation fields may be regarded as particular examples of this kind. In this section, we propose a general framework for deriving the connections between the overall moduli of two-phase composites. The formulation is not limited to the context of elasticity. For general purposes, let us characterize the physical behavior for the phases (designated as 1 and 2) as

$$\begin{cases} \boldsymbol{U}_i = \boldsymbol{P}_i \boldsymbol{X}_i + \boldsymbol{Q}_i \boldsymbol{Y}_i, \\ \boldsymbol{V}_i = \boldsymbol{Q}_i^{\mathrm{T}} \boldsymbol{X}_i + \boldsymbol{R}_i \boldsymbol{Y}_i, \end{cases} : \qquad i = 1, 2, \tag{41}$$

where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$ and $\boldsymbol{R} = \boldsymbol{R}^{\mathrm{T}}$, $\boldsymbol{U}$ and $\boldsymbol{X}$ are $(n-r) \times 1$ matrices, and $\boldsymbol{V}$ and $\boldsymbol{Y}$ are $r \times 1$ matrices. Suppose $\boldsymbol{Y}$ represents the uniform local field quantities throughout the medium, i.e. $\boldsymbol{Y}_1 = \boldsymbol{Y}_2 = \boldsymbol{Y}^u =$ constant matrix, which means that there exist r uniform fields among the n field quantities.

If, in addition, the local variables fulfil certain field equations (for instance, equilibrium equations in elasticity or divergence equations in dielectric problems) so that the average fields over the representative volume Ω are equal to the remote applied loading, namely $\bar{\boldsymbol{X}} = \boldsymbol{X}^{\infty}$ and $\bar{\boldsymbol{Y}} = \boldsymbol{Y}^{u}$, then the overall moduli are necessarily connected by relations of type

$$\bar{\boldsymbol{U}} = \boldsymbol{P}^{*}\boldsymbol{X}^{\infty} + \boldsymbol{Q}^{*}\boldsymbol{Y}^{u},$$
$$\bar{\boldsymbol{V}} = \boldsymbol{Q}^{*\mathrm{T}}\boldsymbol{X}^{\infty} + \boldsymbol{R}^{*}\boldsymbol{Y}^{u}, \tag{42}$$

where

$$\bar{M} = \frac{1}{\Omega}\int_{\Omega} M \,\mathrm{d}\Omega, \quad \boldsymbol{M} = \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{X}, \boldsymbol{Y}. \tag{43}$$

Now taking an average of the field variables and comparing with (42), in analogy to (7) one finds

$$\Delta\boldsymbol{P}^{*}\boldsymbol{X}^{\infty} + \Delta\boldsymbol{Q}^{*}\boldsymbol{Y}^{u} = c\left\{\Delta\boldsymbol{P}\langle\boldsymbol{X}\rangle + \Delta\boldsymbol{Q}\boldsymbol{Y}^{u}\right\},$$
$$\Delta\boldsymbol{Q}^{*\mathrm{T}}\boldsymbol{X}^{\infty} + \Delta\boldsymbol{R}^{*}\boldsymbol{Y}^{u} = c\left\{\Delta\boldsymbol{Q}\langle\boldsymbol{X}\rangle + \Delta\boldsymbol{R}\boldsymbol{Y}^{u}\right\}, \tag{44}$$

where

$$\Delta\boldsymbol{P}^{*} = \boldsymbol{P}^{*} - \boldsymbol{P}_1, \quad \Delta\boldsymbol{Q}^{*} = \boldsymbol{Q}^{*} - \boldsymbol{Q}_1, \quad \Delta\boldsymbol{R}^{*} = \boldsymbol{R}^{*} - \boldsymbol{R}_1,$$
$$\Delta\boldsymbol{P} = \boldsymbol{P}_2 - \boldsymbol{P}_1, \quad \Delta\boldsymbol{Q} = \boldsymbol{Q}_2 - \boldsymbol{Q}_1, \quad \Delta\boldsymbol{R} = \boldsymbol{R}_2 - \boldsymbol{R}_1,$$
$$c = \Omega_2/\Omega_1, \quad \Omega = \Omega_1 + \Omega_2, \quad \langle\boldsymbol{X}\rangle = \frac{1}{\Omega}\int_{\Omega_2} \boldsymbol{X}\,\mathrm{d}\Omega. \tag{45}$$

Since $\boldsymbol{X}^{\infty}$ and $\boldsymbol{Y}^{u}$ can be prescribed arbitrarily, by letting $\boldsymbol{Y}^{u} = 0$ or $\boldsymbol{X}^{\infty} = 0$ separately, we find the connections between the overall moduli

$$\Delta\boldsymbol{Q}^{*} = \Delta\boldsymbol{P}^{*}(\Delta\boldsymbol{P})^{-1}\Delta\boldsymbol{Q},$$
$$\Delta\boldsymbol{R}^{*} - c\Delta\boldsymbol{R} = \Delta\boldsymbol{Q}^{\mathrm{T}}(\Delta\boldsymbol{P})^{-1}[\Delta\boldsymbol{P}^{*} - c\Delta\boldsymbol{P}](\Delta\boldsymbol{P})^{-1}\Delta\boldsymbol{Q}, \tag{46}$$

provided that $\Delta\boldsymbol{P}$ is invertible. Again, when $\Delta\boldsymbol{P}$ is singular the approach outlined in Section 3 can be employed.

6. Closure

For a cylindrical body in which the material properties can be represented by the characterization formulae (4), we show that its effective moduli follow the constraints similar to those for two-phase fibrous media (Hill, 1964). In fact, it can be shown that (4) is the most general characterization for which such connections can be established. Eqn (4) permits us to characterize a domain that is more general than a two- or multi-phase medium or even a pointwise varying material, but in principle they cannot depict a general three-phase material. However, for a three-phase material in which the third phase is itself a composite of the first two materials, or the family of three-phase materials that could be characterized by (4), then the results (9) and (10) still hold. The connections provide relationships between the moduli in the axial direction and those relating to the off-axial direction. In general at most six conditions can be obtained and thus, only 15 out of a total of 21 constants are independent. In the case that the matrix $\Delta\hat{S}$ is singular, additional connections between the effective constants may be found. We mention, however, that the exact result of the bulk modulus found by Hill (1963) for composites with identical shear rigidities, which corresponds to the case that $\Delta\hat{S}$ is singular, cannot be derived from our connections, since the present connections provide exact relations between the axial moduli and off-axial moduli, while Hill's result determines a unique solution to bulk modulus in the transverse direction. In plane elasticity, Zheng and Hwang (1996, 1997) recently found that the effective tensors for a medium containing cavities or inhomogeneities are independent of some phase material parameters. In a companion work, we (Zheng and Chen, 1999) show that the reduced dependence of the phase moduli and connections obtained in the present work are complementary to each other. Thus, it is likely that the effective constants in the axial direction could be irrelevant to some of the phase parameters in the transverse plane. Finally, we remark that, in addition to the various physical contexts mentioned in Section 5, the present framework can be applied to polycrystals (see, for example, Schulgasser 1987; Chen, 1994).

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